# (Electrical) equivalent circuits for microresonators

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This tutorial covers how to develop electrical equivalent circuits for micromechanical resonators. After reading this you should be familiar with terms like effective mass or motional resistance – and some other RF-MEMS[1] jargon. This tutorial is started with an example analysis of a beam resonator. After developing an equivalent circuit for it, a more general approach to electrical equivalents given.

Figure 1 shows a half wavelength beam resonator developed for reference oscillator applications [1]. Before going into more detailed analysis, we make a few qualitative observations. First, the beam ends move in the opposite direction and the beam is acnored from the middle which is the vibrational nodal point. Secondly, the beam is actuated symmetrically using both electrodes and is listened from the center anchor making the resonator a two port device. Applying voltage over the resonator results in ac-current flowing through the device. This current can be divided into two parts: "normal" current $i_{ac}$ due capacitive current path between beam and electrodes and motional current $i_{mot}$ due to motion of the beam end.

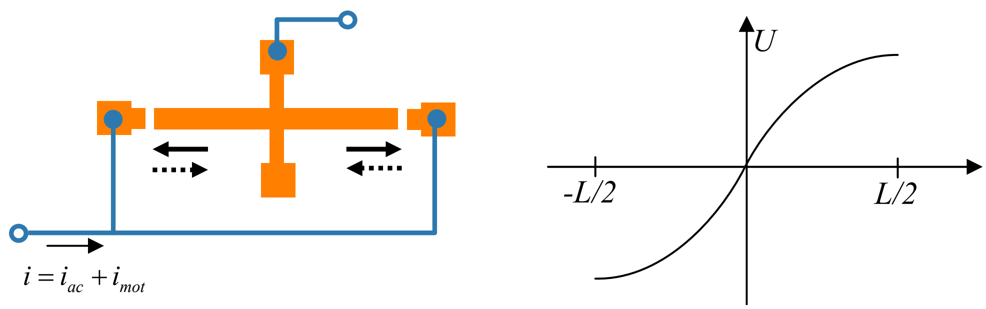


***Figure 1.*** *Schematic of longitudinal mode beam resonator and the vibrational displacement (mode shape) [1].*

## Vibration mode

To analyze the beam resonator in Figure 1, we start with Hooke's equation for stress and strain

$$T = YS, \tag{1}$$

where $T$ is stress, $Y$ is Young's modulus, and $S$ is the strain due to the beam displacement $U$ given by $S = \frac{\partial U}{\partial z}$. The force acting on a small beam segment $\Delta z$ is

$$F = A(T(z+\Delta z) - T(z)) = A\frac{T(z+\Delta z) - T(z)}{\Delta z}\Delta z \approx A\frac{\partial T}{\partial z}\Delta z, \tag{2}$$

where $A$ is the beam cross sectional area. On the other hand, Newton's equation for the segment is

$$F = m\frac{\partial^2 U}{\partial t^2} = \rho A\frac{\partial^2 U}{\partial t^2}\Delta z. \tag{3}$$

[1] Radio Frequency Microelectromechanical Systems

Combining Equations (1), (2), and (3) gives

$$\rho A \frac{\partial^2 U}{\partial t^2} = YA \frac{\partial^2 U}{\partial z^2}. \tag{4}$$

This is a wave equation for a longitudinal motion in a beam. At this point we make a wild guess for the solution and try [2]

$$U = (ae^{jkz} + be^{-jkz})e^{j\omega_0 t} \tag{5}$$

Turns out that this just happens to be the solution provided that

$$\rho\omega_0^2 = Yk^2. \tag{6}$$

The boundary conditions are given by the requirement that there is no stress and no stress gradient on the free end surfaces. Therefore we have

$$\begin{array}{l} T = Y\frac{\partial U}{\partial z} = 0 \text{ and} \\ \frac{\partial T}{\partial z} = 0 \end{array} \tag{7}$$

at the beam ends ($z = \pm L/2$). Thus, the solution is

$$U = \sin(kz)e^{j\omega_0 t}, \tag{8}$$

where $k = \pi/L$ [3].

## Lumped model for forced vibrations

Lets now complicate the situation a bit by adding damping and excitation to the model. Equation (4) now becomes

$$\rho A \frac{\partial^2 U}{\partial t^2} + bA\frac{\partial U}{\partial t} - YA\frac{\partial^2 U}{\partial z^2} = F(z,t), \tag{9}$$

where $b$ is the damping and $F(z,t)$ is the time harmonic electrostatic force at the beam ends. We'll come back to this a bit later but for now we'll write it simply as

$$F(z,t) = \frac{f(t)}{2}(\delta(z - L/2) - \delta(z + L/2)), \tag{10}$$

where $\delta$ is the delta-function. Now it would be really nice if we could somehow use the hard work done in the previous section to obtain a solution to Equation (9). Equation (8) does not work right away but we'll take it is as a starting point and assume that the *mode shape* remains the same and only the time behavior changes. We thus write the solution to Equation (4) as

$$U(z,t) = x(t)\sin kz, \tag{11}$$

where $x$ is the motion of the beam tip. Substituting Equation (11) to (9) leads to

$$\rho A\frac{\partial^2 x}{\partial t^2}\sin kz + bA\frac{\partial x}{\partial t}\sin kz + YAk^2 x \sin kz = F(z,t). \tag{12}$$

[2] A more systematic way would have been to try the separation of variables with $U(z,t) = \Gamma(z)\Theta(t)$.

[3] Ok, there really is infinite number of solutions. We just chose the first one (the one with the lowest $k$).

Next we'll multiply Equation (12) with the mode shape $\sin kz$ and integrate over the beam length. Noting that $\int_{-L/2}^{L/2} \sin^2 kz\mathrm{d}z = L/2$ and that $\int_{-L}^{L} F(z,t) \sin kz\mathrm{d}z = f(t)$, this leads to

$$\frac{\rho AL}{2}\frac{\partial^2 x}{\partial t^2} + \frac{bAL}{2}\frac{\partial x}{\partial t} + \frac{YAk^2L}{2}x = f(t). \tag{13}$$

Recognizing the effective mass, damping coefficient, and the spring constant as

$$\begin{aligned} &M = \rho AL/2 \\ &\gamma = bAL/2 \\ &K = YAk^2L/2 = \pi^2 YA/2L, \end{aligned} \tag{14}$$

leads to familiar equation for forced vibrations of damped resonator given by

$$M\frac{\partial^2 x}{\partial t^2} + \gamma\frac{\partial x}{\partial t} + Kx = f(t). \tag{15}$$

Since, we know the solution to Equation (15), we stop here.

## Capacitive excitation

To actuate our beam, we use electrostatic force. This requires a large dc-voltage $U_{dc}$ over the narrow gap between the beam end and the electrode superposed by an ac-voltage $v(t)$ at the actuation frequency $f$. The energy stored in the parallel plate capacitor $C$ formed by the beam end and the electrode is

$$E = \frac{1}{2}Cu^2 = \frac{1}{2}C\left(u_{dc}^2 + 2u_{ac}u_{dc} + u_{ac}^2\right), \tag{16}$$

Thus, we have force at three frequencies: a dc-force, force at excitation frequency $f$ due to cross term $2u_{ac}u_{dc}$, and force at twice the excitation frequency due to square term $u_{ac}^2$. The last term is assumed small so we'll ignore it. We'll also forget about the dc-term as we are trying to model a resonator. This implies that the model is valid near the resonance if the quality factor is high.

The plate capacitance is

$$C = \varepsilon\frac{A_{el}}{d-x}, \tag{17}$$

where $\varepsilon$ is the permitivity of free space, $A_{el}$ is the electrode area, $d$ is the initial electrode gap, and $x$ is again the movement of the beam tip. The force is obtained from

$$f = \frac{\partial E}{\partial x} = \frac{1}{2}u^2\frac{\partial C}{\partial x} = u^2\varepsilon\frac{A_{el}}{2(d-x)^2}. \tag{18}$$

Equation (18) is complicated as $1/(d-x)^2$ term is nonlinear. Since we do not like complications, we'll assume that that $x << d$ to linearize Equation (18). Putting all the approximations together ($u^2 \approx 2u_{ac}u_{dc}$ and $x << d$), we get

$$f = u_{ac}u_{dc}\varepsilon\frac{A_{el}}{d^2} = \eta u_{ac}, \tag{19}$$

where we have defined a new variable, the *electromechanical transduction factor* $\eta = u_{dc}\varepsilon\frac{A_{el}}{d^2}$.

## Resonator current

The current through the resonator is

$$i = \frac{\partial Cu}{\partial t} = C\frac{\partial u}{\partial t} + u\frac{\partial C}{\partial t}. \tag{20}$$

With the approximations made in the previous section, Equation (20) becomes

$$i = C_0\frac{\partial u_{ac}}{\partial t} + \eta\frac{\partial x}{\partial t} = i_{ac} + i_{mot}. \tag{21}$$

We recognize the first term as the normal ac-current through capacitor and second term is *motional current* $i_{mot} = \eta\frac{\partial x}{\partial t}$ due to time-varying capacitance.

## Electrical equivalent circuit

We are now ready to develop an equivalent circuit for our beam resonator. We start by substituting $v = \frac{\partial x}{\partial t} = i_{mot}/\eta$ into Equation (15) giving

$$\frac{M}{\eta}\frac{\partial i_{mot}}{\partial t} + \frac{\gamma}{\eta}i_{mot} + \frac{K}{\eta}\int i_{mot}\mathrm{d}t = f(t). \tag{22}$$

Next, remembering that $f = \eta u_{ac}$ from Equation (19) we write Equation (22) as

$$\frac{M}{\eta^2}\frac{\partial i_{mot}}{\partial t} + \frac{\gamma}{\eta^2}i_{mot} + \frac{K}{\eta^2}\int i_{mot}\mathrm{d}t = u_{ac}. \tag{23}$$

By defining motional resistance, motional capacitance, and motional inductance as

$$\begin{aligned} R_m &= \gamma/\eta^2 = \sqrt{KM}/Q\eta^2, \\ C_m &= \eta^2/K, \text{ and} \\ L_m &= M/\eta^2 \end{aligned} \tag{24}$$

Equation (23) becomes

$$L_m\frac{\partial i_{mot}}{\partial t} + R_m i_{mot} + \frac{1}{C_m}\int i_{mot}\mathrm{d}t = u_{ac}. \tag{25}$$

This is a series RLC-circuit that relates motional current to actuation voltage. The whole beam resonator can be represented by the electrical equivalent circuit is shown in Figure 2. The motional arm is represented by the series RLC-network and the capacitance $C_0$ represents the non-motional current path.

## General electrical equivalent representation

We have shown that our beam resonator can, quite naturally in fact, be represented by a series RLC resonator. This, however, is not the only possible representation and may not even be the best depending on the analysis and application. In this section a more general approach is given and two analogies, voltage and current, are derived.

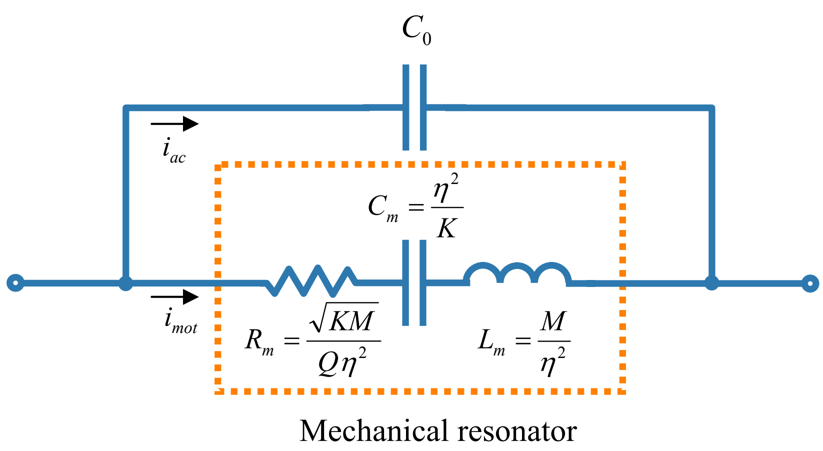


***Figure 2.*** *Electrical equivalent circuit for microresonator.*

*Current analogue*

Writing Equation (15) in terms of velocity $v$ gives

$$M\frac{\partial v}{\partial t}+\gamma v+K\int v\mathrm{d}t=f. \tag{26}$$

The current analogue is obtained by setting

$$\begin{aligned} v &= i_{vel} \text{ and} \\ f &= u_{force} \end{aligned} \tag{27}$$

The electrical equivalent is then

$$L_m\frac{\partial i_{vel}}{\partial t}+R_m i_{vel}+\frac{1}{C_m}\int i_{vel}\mathrm{d}t=u_{force}, \tag{28}$$

where

$$\begin{aligned} R_m &= \gamma, \\ C_m &= 1/K, \text{ and} \\ L_m &= M. \end{aligned} \tag{29}$$

Again this is series LRC resonator. For a complete equivalent we need to the relate the mechanical current and voltage to real current and voltage. In the case of capacitive coupling, the relationship is

$$\begin{aligned} i_{circ} &= \eta i_{vel} \text{ and} \\ u_{circ} &= u_{force}/\eta. \end{aligned} \tag{30}$$

Equation (30) is a transformer with ratio of $\eta:1$. The whole circuit is shown in Figure 3.

*Voltage analogue*

The voltage analogue is obtained from Equation (26) by setting

$$\begin{aligned} v &= u_{vel} \text{ and} \\ f &= i_{force} \end{aligned} \tag{31}$$

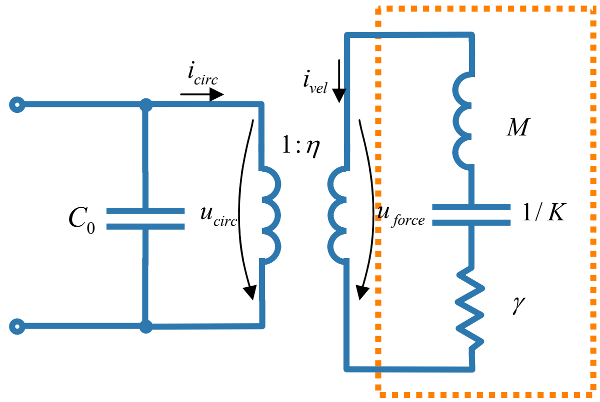


***Figure 3.*** *Current analogue for a mechanical resonator ($v = i_{vel}$ and $f = u_{force}$).*

The electrical equivalent is then

$$C_m \frac{\partial u_{vel}}{\partial t} + 1/R_m u_{vel} + \frac{1}{L_m} \int u_{vel} \mathrm{d}t = i_{force}, \tag{32}$$

where

$$\begin{aligned} R_m &= 1/\gamma, \\ C_m &= M, \text{ and} \\ L_m &= 1/K. \end{aligned} \tag{33}$$

This is a parallel LRC resonator. Assuming again capacitive coupling, the relationship between mechanical and real current and voltages is given by

$$\begin{aligned} i_{circ} &= \eta u_{vel} \text{ and} \\ u_{circ} &= i_{force}/\eta. \end{aligned} \tag{34}$$

Equation (34) is a gyrator – a circuit element that is readily available in simulators but requires active elements for hardware implementations. The whole circuit is shown in Figure 4.

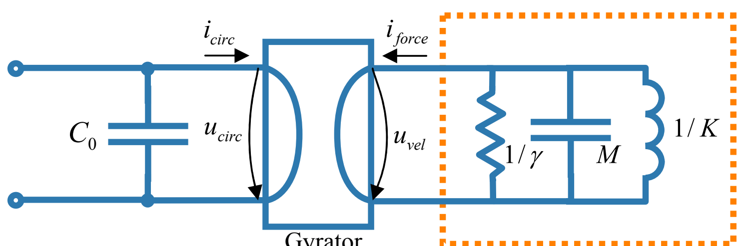


***Figure 4.*** *Voltage analogue for a mechanical resonator ($v = u_{vel}$ and $f = i_{force}$).*

## References

[1] T. Mattila, J. Kiihamäki, T. Lamminmäki, O. Jaakkola, P. Rantakari, A. Oja, H. Seppä, H. Kattelus, and I. Tittonen, "A 12 MHz micromechanical bulk acoustic mode oscillator" Sensors and Actuators A, Vol. 101, no. 1-2, pp. 1-9, Sep. 2002.