MEMS Tutorial:
Mechanical noise in microelectromechanical systems

Mircomachining has enabled manufacturing of cheap and reliable minituarized sensors. For variety of reasons, it would be interesting to further scale down the devices. For example, it may be of interest to obtain higher resonance frequency or simply to lower the cost by incorporating more devices on a silicon wafer. However, the smaller devices the lower is the signal to noise ratio [1, 2]. Intuitively this may be understood by noting that ratio of mechanical to thermal energy E/kT goes down as the device mass is reduced. This tutorial covers the derivation and analysis of noise in mechanical devices.

Derivation of Johnson-Nyguist noise

It may seem strange to start the discussion of mechanical noise with the derivation of Johnson-Nyquist noise as mechanical vibrations and electron movement seem to have little in common. However, as the origin of noise in both cases is dissipation, it should no surprise that the resulting expressions for the noise are very similar. Thus, reviewing Johnson-Nyguist noise serves to reassure the reader who probably is already familiar with thermal noise in resistors: once we have derived the correct expression for the resistor noise voltage generator ($\overline{u_n^2} = 4kTR$), a similar derivation for the case mechanical dissipation will be easier to accept. The derivation done here is a bit mathematical. To those who enjoy good argumentation without too many equations, you may want to obtain a copy of the original article by Nyguist [3] as it is still one of the most readable derivations for the resistor noise.

We'll begin by looking at the series RLC resonator shown in Figure 1. Associated with the resistor is some unknown thermal noise generator u_n. Due to this voltage there is energy stored in the inductor and capacitor. We also note that the circuit is fully described with two variables: current and voltage (or magnetic and electric). From the equipartititon theorem we know that there is thermal energy $\frac{1}{2}kT$ associated with the every variable. We now have a road map for the derivation: calculate the energy stored in the inductor due to voltage u_n and equate this with $\frac{1}{2}kT$.

The current through the RLC network due to noise voltage v_n is

$$i_n = \frac{u_n}{R + sL + 1/sC}, \tag{1}$$

where $s = j\omega$ as usual. The magnitude of the mean square current is

$$\overline{i_n^2} = \frac{\overline{u_n^2}}{R^2 + (\omega L - 1/\omega C)^2}. \tag{2}$$

We'll rewrite Equation (2) in terms of resonance frequency $\omega_0 = 1/\sqrt{LC}$ and quality factor $Q = \omega_0 L/R$ as

$$\overline{i_n^2} = \frac{1}{R^2}\frac{\overline{u_n^2}}{1 + Q^2(\omega/\omega_0 - \omega_0/\omega)^2}. \tag{3}$$

Figure 1. *RLC resonator with noise generator v_n*

The energy stored in the inductor in a frequency interval is $\mathrm{d}E = \frac{1}{2}L\overline{i_n^2}\mathrm{d}f$. Thus, the total energy stored in the inductor is

$$E = \frac{1}{2}L\int_0^\infty \overline{i_n^2}\mathrm{d}f. \tag{4}$$

Substituting Equation (3) into Equation (4) gives

$$E = \frac{1}{4\pi R}\int_0^\infty \frac{\overline{u_n^2}Q\mathrm{d}(f/f_0)}{1+Q^2(f/f_0-f_0/f)^2}. \tag{5}$$

In order to evaluate the integral in Equation (5), we assume that the quality factor is high so that essentially all the energy is confined near the resonance. We can thus assume that within this small frequency range the voltage generator is approximately constant and write $\overline{u_n^2} \approx \overline{u_n^2(f_0)}$ giving

$$E = \frac{\overline{u_n^2(f_0)}}{4\pi R}\int_0^\infty \frac{Q\mathrm{d}(f/f_0)}{1+Q^2(f/f_0-f_0/f)^2}. \tag{6}$$

Evaluation of the integral is obviously left for the reader as an exercise [1]. The result is

$$E = \frac{\overline{u_n^2(f_0)}}{8R}. \tag{7}$$

Equating (7) with the thermal energy $\frac{1}{2}kT$ gives

$$\overline{u_n^2(f_0)} = 4kTR. \tag{8}$$

Since the result does not depend on frequency, it is valid for all frequencies and we have

$$\overline{u_n^2} = 4kTR. \tag{9}$$

Derivation of mechanical noise

We'll now proceed to the problem of noise in a mechanical resonator shown in Figure 2. We note that the system is fully described by two variables: velocity and position (or kinetic and potential energy). Associated with damping, there is some force noise generator F_n for which we wish to derive an expression.

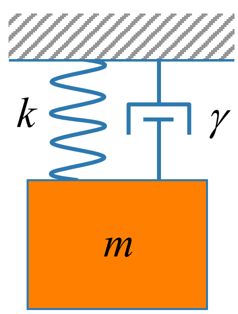

Figure 2. *Mechanical resonator represented with mass M, spring K, and dash pot damper γ.*

The equation of motion for the system is

$$M\frac{\partial^2 x}{\partial t^2} + \gamma\frac{\partial x}{\partial t} + Kx = F_n. \tag{10}$$

[1]Hint: Start by writing $\frac{f}{f_0} = e^x$ and proceed from there.

We may write Equation (10) in terms of velocity v by noting that $v = \frac{\partial x}{\partial t} = sx$ giving

$$sMv + \gamma v + \frac{K}{s}v = F_n. \tag{11}$$

Thus, the mean square velocity due to noise generator F_n is

$$\overline{v_n^2} = \frac{\overline{F_n^2}}{\gamma^2 + (\omega M - K/\omega)^2}. \tag{12}$$

Equation (12) can be rewritten in terms of resonance frequency $\omega_0 = \sqrt{K/M}$ and quality factor $Q = \omega_0 M/\gamma$ as

$$\overline{v_n^2} = \frac{1}{\gamma^2}\frac{\overline{F_n^2}}{1 + Q^2(\omega/\omega_0 - \omega_0/\omega)^2}. \tag{13}$$

The kinetic energy stored in the resonator is

$$E = \frac{1}{2}m\overline{v_n^2} = \frac{1}{4\pi\gamma}\int_0^\infty \frac{\overline{F_n^2}Q\mathrm{d}(f/f_0)}{1 + Q^2(f/f_0 - f_0/f)^2}, \tag{14}$$

which is the same as Equation (5). Thus, the force noise generator is

$$\overline{F_n^2} = 4kT\gamma. \tag{15}$$

Mechanical noise in electrical equivalent circuit

Another way to derive force noise generator is to use equivalent circuit analysis. We start by substituting $\frac{\partial x}{\partial t} = i$ into Equation (10) giving

$$M\frac{\partial i}{\partial t} + \gamma i + K\int i\mathrm{d}t = f(t). \tag{16}$$

Next, defining $f = u$ we rewrite Equation (16) as

$$M\frac{\partial i}{\partial t} + \gamma i + K\int i\mathrm{d}t = u. \tag{17}$$

By defining motional resistance, motional capacitance, and motional inductance as

$$\begin{aligned} &R_m = \gamma = \sqrt{KM}/Q, \\ &C_m = 1/K, \text{ and} \\ &L_m = M \end{aligned} \tag{18}$$

Equation (17) becomes

$$L_m\frac{\partial i}{\partial t} + R_m i + \frac{1}{C_m}\int i\mathrm{d}t = u. \tag{19}$$

With Laplace notation this is

$$sL_m i + R_m i + \frac{1}{sC_m}i = u, \tag{20}$$

which is same as Equation (1) for series RLC circuit. The noise voltage is

$$\overline{u_n^2} = 4kTR_m. \tag{21}$$

Using the definitions $f = u$ and $R_m = \gamma$, the mechanical noise is then

$$\overline{f_n^2} = 4kT\gamma, \tag{22}$$

which is the same as Equation (15).

Noise in accelerometers

We'll exemplify what we have learned by looking at noise in a micromechanical accelerometer. A structure of a typical MEMS accelerometer is shown in Figure 3. The device dimensions are 1 mm·1 mm·0.2 mm and the mass and spring constant are $4.4 \cdot 10^{-7}$ kg and 0.25 N/m respectively. The noise spectrum is obtained from Equations (10) and (15) as

$$\overline{x_n^2} = \frac{4kT\gamma}{\gamma^2\omega^2 + (K - M\omega^2)^2}. \tag{23}$$

The rms noise $< x_n >= \sqrt{\overline{x_n^2}}$ is plotted in Figure 4 for quality factor values of $Q = 1$, $Q = 2$, and $Q = 3$.

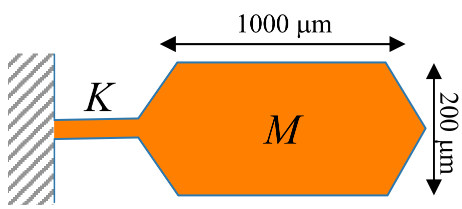

Figure 3. *A schematic of MEMS accelerometer.*

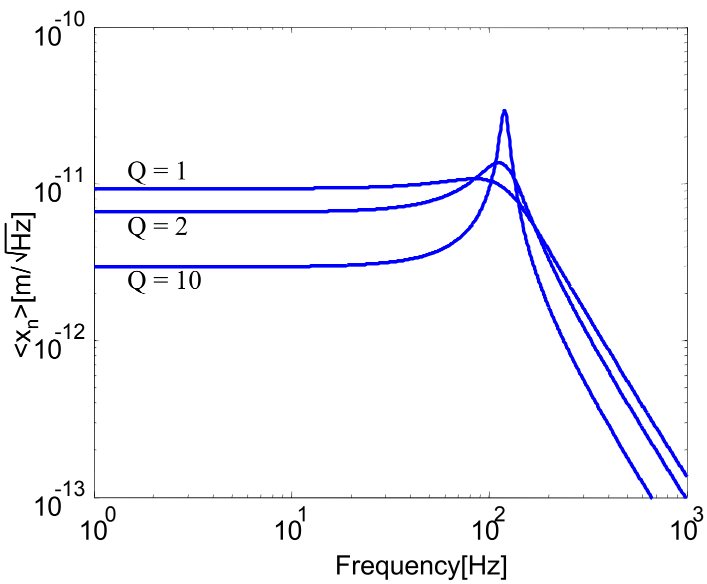

Figure 4. *The calculated accelerometer noise spectrum.*

The accelerometers are used to detect motion below the resonance frequency where

$$\overline{x_n^2} \approx \frac{4kT\gamma}{K^2} \tag{24}$$

and the RMS noise is

$$< x_n > \approx \frac{\sqrt{4kT\gamma}}{K}. \tag{25}$$

The force due to acceleration is $F = ma$ and below the resonance the movement of the proof mass is

$$x \approx \frac{ma}{K}. \tag{26}$$

The accelerometer noise is then

$$< a_n > \approx \sqrt{\frac{4kT\omega_0}{mQ}}. \tag{27}$$

For example, Equation (27) gives $< a_n >= 0.38\ \mu\mathrm{g}/\sqrt{\mathrm{Hz}}$ for our example accelerometer $Q = 2$. Looking at Equation (27), it is observed that the signal to noise ratio can be increased by i. increasing the mass, ii. increasing the quality factor, and iii. reducing the bandwidth. The quality factor, however, cannot be increased too much, at least not without feedback control, as high Q will result in excessive ringing at the resonance frequency. As the operational bandwidth is usually fixed, the designer is left with increasing the proof mass as the only method to increase signal to noise ratio. This may partially explain why there are so few "nanomechanical" accelerometers on the market [2].

The theory can be put into perspective by comparing two commercial accelerometers. A bulk micromachined accelerometer (SCA610 from VTI) has a noise of $< a_n >= 30\ \mu\mathrm{g}/\sqrt{\mathrm{Hz}}$, while a surface micromachined accelerometer (ADXL150 from Analog Devices) that has much smaller mass has noise of $< a_n >= 1\ \mathrm{mg}/\sqrt{\mathrm{Hz}}$. While these numbers include noise from circuitry, they do illustrate the need for large mass to obtain low noise.

References

[1] T. B. Gabrielson, "Mechanical-thermal noise in micromachined acoustic and vibration sensors", IEEE Trans. Electron Dev., vol. 40 , no. 5, pp. 903-909, 1993.

[2] A. N. Cleland and M. L. Roukes, "Noise processes in nanomechanical resonators", J. Appl. Phys., vol. 92, no. 5, pp. 2758-2769, 2002

[3] H. Nyguist, "Thermal Agitation of Electric Charge in Conductors", Phys. Rev., vol. 32, pp. 110-113, 1928.

[2] By few we mean zero and zero is a nanosmall number.